Generic Rank of a Family of Elliptic Curves

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Motivation

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One reason: We didn't want to do the exercises.

Definitions

Definition

A family of elliptic curves $\mathcal{E}$ is given by the equation

$$\mathcal{E} : y^2 + a_1(T)xy + a_3(T)y = x^3 + a_2(T)x^2 + a_4(T)x + a_6(T)$$

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- Will assume $a_1 = a_3 = 0$ with $\deg a_i \leq 2$ for i even.

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Theorem (Silverman)

We have $\mathrm{rk}(\mathcal{E}_t) \geq \mathrm{rk}(\mathcal{E}(\mathbb{Q}(T)))$ for all but finitely many $t \in \mathbb{Q}$.

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  Q3: Suppose there is no elliptic curve E over $\mathbb{Q}$ such that $\mathcal{E} \cong E \times \mathbb{P}^1$.

  Is $\mathcal{E}(\mathbb{Q})$ Zariski dense?

Computing the generic rank

Conjecture (Nagao)

The rank of $\mathcal{E}$ over $\mathbb{Q}(T)$ is

$$r_{\mathcal{E}} = \lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -A_{\mathcal{E}}(p)\log p,$$

where p runs through all primes $p \le X$ and

$$A_{\mathcal{E}}(p) := \frac{1}{p}\sum_{t=0}^{p-1} a_{\mathcal{E}_t}(p),$$

where $a_{\mathcal{E}_t}(p) = p + 1 - \#\mathcal{E}_t(\mathbb{F}_p)$.

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- ⋆ with no multiplicative reduction except possibly at infinity.

Problems we considered

Elliptic surfaces with multiplicative reduction at finite places

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- Density of rational points

More on Nagao's conjecture

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Assume $\mathcal{E}$ is not constant. Then the generic rank is

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\leq X} \frac{-\log p}{p} \sum_{t=0}^{p-1} a_{\mathcal{E}(t)(p)},$$

where $a_{\mathcal{E}(t)(p)}$ is the trace of Frobenius at p of the specialisation at t.

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This is true in the case of rational elliptic surfaces, due to Rosen and Silverman.

An Example of What we Found

Proposition (BBDKPP)

Let $k \in \mathbb{Q}^\times$ and consider the family of elliptic surfaces

$$\mathcal{E}_k : y^2 = x^3 + T^2x + kT^2.$$

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$$\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) \le 1,$$

with equality if and only if $k \in \pm \left(\mathbb{Q}^\times\right)^2$.

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with equality if and only if $k \in \pm \left(\mathbb{Q}^\times\right)^2$. Moreover, the generating section is

$$\begin{aligned} (0, \sqrt{k}T) &\quad \text{if } k \text{ is a square;} \\ (-k, \sqrt{(-k)^3}) &\quad \text{if } -k \text{ is a square.} \end{aligned}$$

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Shioda-Tate Formula

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In our case

Let $\mathcal{E}_k : y^2 = x^3 + T^2x + kT^2$. Then $\Delta(\mathcal{E}_k) = -16T^4(4T^2 + 27k^2)$,
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- At T, we have type IV $(m_v = 3)$;
- At the linear factors of $(4T^2 + 27k^2)$, we have type I_1 $(m_v = 1)$;
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So $\operatorname{rank} \mathcal{E}_k(\overline{\mathbb{Q}}(T)) = 10 - 2 - (3-1) - 2(1-1) - (5-1) = 2$.

Classification when $a_2 = 0$

Theorem (BBDKPP)

Consider the non-isotrivial elliptic surface

$$\mathcal{E} : y^2 = x^3 + a_4(T)x + a_6(T),$$

with $\deg a_i \leq 2$ such that there are exactly two fibres of multiplicative reduction over $\overline{\mathbb{Q}}$.

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with $\deg a_i \leq 2$ such that there are exactly two fibres of multiplicative reduction over $\overline{\mathbb{Q}}$. Then $\mathcal{E}$ belongs to one of the following families:

- $y^2 = x^3 + kx + T$ with $k \in \mathbb{Q}^\times$;
- $y^2 = x^3 + (aT + b)x + (aT^2 + bT)$ where $a \neq 0$ and $b \neq a^2/27$;
- $y^2 = x^3 + P(T)x + kP(T)$ for some quadratic polynomial P and $k \in \mathbb{Q}^\times$ such that $4P(T) + 27k^2$ is nonsquare in $\overline{\mathbb{Q}}[T]$.

Example

The isotrivial elliptic surface

$$\mathcal{E} : y^2 = x^3 + T$$

has $\mathrm{rank}(\mathcal{E}(\mathbb{Q}(T))) = 0$.

However, it has infinite subfamilies of positive rank. In particular, the subfamily of elliptic curves (given by Nagao)

$$\mathcal{E}_s : y^2 = x^3 + (s^2 - m^3)$$

has generic rank 1 for any fixed $m \in \mathbb{Z} \setminus 0$.

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- Family of constant root number ($W(\mathcal{E}_t) = -1$) for all $t \in \mathbb{Q}$) found by Julie.
- Our method doesn't work since $\deg a_i$ too large. :-(

Open questions and possible future work

- Use known families with constant root number to guess interesting subfamilies of elliptic curves with high rank?
- Generic rank when $\deg a_i$ is high

Thank you!

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